# Chapter 5

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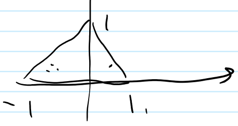
1) $E(X_1 + X_2) \neq E(X_1) + E(X_2)$



$E(X_2) = \int_{-1} \int_0^{x+1} y \, dy \, dx = \frac{1}{3}$

2) $\operatorname{Var} X = E\left(\left(X - E(X)\right)^2\right) = E(X^2) - \left(E(X)\right)^2$

4) $X, Y$ - same distribution

$XY \equiv 0 \qquad E(X) = E(Y) = 0$.

$P\{X=-1, Y=-1\} = 0 \neq P(X=-1) \cdot P(Y=-1)$.

6a) $\int |X| \, dP < \infty$

Need: $\int_{\{|X|>n\}} X \, dP \to 0$

$\left|\int X \, dP\right| \le \int |X| \, dP$

$Y_n = \begin{cases} |X|, & |X| \le n \\ 0, & |X| > n \end{cases}$

$Y_n \nearrow$, $Y_n \ge 0$.

$\int Y_n \, dP \to \int |X| \, dP$ By Monotone Convergence.

$\|$

$\int_{\{|X| \le n\}} |X| \, dP = \int |X| \, dP - \int_{\{|X|>n\}} |X| \, dP$.

15. $P(0 \le X < \infty) = 1$.

a) $\lim_{n \to \infty} \left( n E\left( \frac{1}{X} 1_{[X>n]} \right) \right)$

a) $\lim_{n\to\infty} \left( n E\left( \frac{1}{X} 1_{[X>n]} \right) \right)$

$E\left( \underbrace{\frac{n}{X}}_{\le 1} 1_{[X>n]} \right) \le E\left( 1_{[X>n]} \right) = P(X>n) \to 0.$

Don't know if $E(|X|)<\infty$.

$\bigcap \{X>n\} = \{X=\infty\}$

b) $\lim_{n\to\infty} E\left( \frac{1}{nX} 1_{[X>\frac{1}{n}]} \right)$

$\forall \omega: X(\omega)>0 \quad \frac{1}{nX(\omega)} \xrightarrow[n\to\infty]{} 0.$

$Y_n(\omega) = \frac{1}{nX} 1_{[X>\frac{1}{n}]} \qquad 0 \le Y_n(\omega) \le 1$

$Y_n \to 0$ a.s. $\qquad X(\omega)=0 \Rightarrow Y_n(\omega)=0$

By Monotone Convergence Theorem, $\qquad \forall n.$

$E(Y_n) \to 0$

21. $P(s) = \sum_{k=0}^{\infty} p_k s^k \qquad \sum p_k = 1, \ p_k \ge 0$

$P(\{k\}) = p_k.$

$P(s) = E(s^X)$

$\lim_{h\to 0} \frac{P(s+h) - P(s)}{h} = \lim \frac{E((s+h)^X) - E(s^X)}{h} =$

$$\lim_{h\to 0} \frac{P(s+h)-P(s)}{h} = \lim_{h\to 0} \frac{E((s+h)^X) - E(s^X)}{h} =$$

$$\lim_{h\to 0} E\left(\frac{(s+h)^X - s^X}{h}\right)$$

$$\lim_{h\to 0} \frac{(s+h)^X - s^X}{h} = X s^{X-1}$$

$$\frac{(s+h)^X - s^X}{h} = (s+h)^{X-1} + (s+h)^{X-2} s + \dots + s^{X-1} \le t^{X-1} \cdot X$$

$s < t < 1$

$s+h < t$

$$E((2s)^{X-1} X) = \sum p_k t^{k-1} k < \infty.$$

Dominated convergence:

$$E\left(\frac{(s+h)^X - s^X}{h}\right) \to E(X s^{X-1}) = \sum_{k=1}^{\infty} p_k k s^{k-1}$$

$$\parallel$$

$$P'(s)$$

$0 \le s \le 1$

$P(1) = \sum p_k = 1.$

$P'(1) = \sum k p_k = E(X) \overset{?}{<} \infty$

if $E(X) < \infty$

36) $X_n \uparrow X$ $\overset{?}{\Rightarrow}$ $X \in L^1.$

$\sup E(X_n)$ ...

$\sup E(X_n) < \infty$ $\quad E(X_n) \to E(X)$.

~~Consider $X_n - X \uparrow 0$~~
~~$X_n - X \le 0$.~~ $\quad$ Monotone convergence $\Rightarrow$ ~~$E(X_n - X)$~~ $\to 0$.

$X_n - X_1 \geq 0$ $\quad X_n - X_1 \uparrow X - X_1$

Monotone convergence $\Rightarrow$ $\quad E(X - X_1) = \lim E(X_n - X_1) = \lim E(X_n) - E(X_1)$

$$E(X) = E(X - X_1) + E(X_1).$$

14. $X, Y$ – independent.

$h : \mathbb{R}^2 \to \mathbb{R}_+$ – non-negative.

$g(x) = E(h(x, Y))$

$$E(g(X)) = E(E(h(X, Y))) =$$

$$\int \underbrace{\left(\int h(x, y)\, P_Y(dy)\right)}_{g(x)} P_X(dx) \overset{\text{indep. Fubbini}}{=}$$

$$\iint h(x, y)\, P(dx, dy) = E(h(X, Y))$$

37) $X = X^+ - X^-$

$$E(X) = E(X^+) - E(X^-)$$

$$Y_n^+ \nearrow X^+ \qquad Y_n^+ = \sum \frac{k}{2^n} \mathbb{1}_{\{\frac{k}{2^n} \leq X < \frac{k+1}{2^n}\}}$$

$$Y_n^- \nearrow X^-$$

$$|X - Y_n| = \begin{cases} X^+ - Y_n^+, & X \geq 0 \\ X^- - Y_n^-, & X < 0. \end{cases}$$